Prof. Dr. M. Günther
Igor Kossaczký, M.Sc.
Winter Term 2014/15

Bergische Universität Wuppertal
Fachbereich C – Mathematik und Naturwissenschaften
Angewandte Mathematik / Numerische Analysis

Lab Exercises for Numerical Analysis and Simulation I: ODEs

Laboratory 3 - Numerical Solution of the Black-Scholes Equation Based on the Method of Lines

Presentation of exercises: 15. - 18. 12.2014

Each working group (1-3 persons) shall present and explain their programmes. Please contact Igor Kossaczký (`kossaczky@math.uni-wuppertal.de`) for a date arrangement.

Consider the Black-Scholes (BS) differential equation

$$H_t = -rSH_s - \frac{1}{2}\sigma^2 S^2 H_{ss} + rH, \qquad t \in [0,T],\ S \in [0, S_{\max}] \tag{BS}$$

with risk-free interest rate r and volatility σ. H is the value of the financial derivative (i.e. a Put-Option in the following) and S the underlying stocks price. The boundary conditions for a European-Put-Option read

$$\begin{aligned} H(0,t) &= \exp(-r(T-t)) \cdot K \qquad \forall\, t \in [0,T] \\ H(S_{\max},t) &= 0 \qquad \forall\, t \in [0,T] \\ H(S,T) &= \max(K-S,0) \qquad \forall\, S \in [0,S_{\max}]. \end{aligned}$$

Thereby, K is the strike and T represents the maturity date. Furthermore, the upper bound $S_{\max} \in \mathbb{R}$ is chosen sufficiently large.

Solve the Black-Scholes differential equation (BS) using a BDF-2 scheme. Proceed as follows:

- Transform equation (BS) via a spatial discretizations in a system of ordinary differential equations of first order. (This is called *Method of Lines.*)
- You obtain an IVP which can immediately be solved with an ordinary integration scheme.

The different steps for the implementation are described below:

- Use a semi-discretization for the stocks price

  $$0 = S_0 < S_1 < S_2 < \ldots < S_N = S_{\max}$$

  with $S_j = j \cdot \Delta S = j \cdot \frac{S_{\max}}{N}$ and $j = 0, \ldots, N$.

- Approximate H_S and H_{SS} with symmetric difference quotients

  $$\begin{aligned} H_S(S_j,t) &= \frac{H(S_{j+1},t) - H(S_{j-1},t)}{2\Delta S}\ , \\ H_{SS}(S_j,t) &= \frac{H(S_{j+1},t) - 2H(S_j,t) + H(S_{j-1},t)}{(\Delta S)^2} \end{aligned}$$

- Define new variables

$$Y_j(t) = H(S_j, t) \qquad \text{for} \qquad j = 0, \dots, N$$

for the implementation. It follows $Y_0(t) = H(S_0, t) = K$ and $Y_N(t) = H(S_N, t) = 0$.

- We get the IVP

$$\begin{aligned} Y'(t) = &- \frac{rN}{2S_{\max}} \operatorname{diag}(S_1, S_2, \dots, S_{N-1}) \cdot \left[A \begin{pmatrix} Y_1 \\ \vdots \\ Y_{N-1} \end{pmatrix} - \begin{pmatrix} K \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right] \\ &- \frac{\sigma^2 N^2}{2S_{\max}^2} \operatorname{diag}(S_1, S_2, \dots, S_{N-1}) \cdot \left[B \begin{pmatrix} Y_1 \\ \vdots \\ Y_{N-1} \end{pmatrix} + \begin{pmatrix} K \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right] + r \begin{pmatrix} Y_1 \\ \vdots \\ Y_{N-1} \end{pmatrix} \end{aligned}$$

with "initial value"

$$Y_j(T) = \max(K - S_j, 0) \qquad \text{for} \qquad j = 0, \dots, N$$

and therefore integrate backwards in time. The matrices A and B read

$$A = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 0 & 1 \\ & & & -1 & 0 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{pmatrix}$$

- Use a BDF-2 scheme to solve the aforementioned system of ODEs of dimension $N - 1$.
- Choose appropriate values for the variables r, σ, S, K and T. You can also normalize them, i.e. set these values as 1.

For further information on the lab exercises, see:
`http://www-num.math.uni-wuppertal.de/en/amna/teaching/lectures/lab-exercises-for-numerical-analysis-and-simulation-i-odes.html`