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Winter Term 2014/15

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# Lab Exercises for Numerical Analysis and Simulation I: ODEs

## Laboratory 2 - Runge-Kutta-Fehlberg

**Presentation of exercises:** 24. - 28. 11.2014

Each working group (1-3 persons) shall present and explain their programmes. Please contact Igor Kossaczký (`kossaczky@math.uni-wuppertal.de`) for a date arrangement.

In this exercise, you should write a MATLAB-routine to solve the initial value problem (IVP)

$$y'(x) = f(x, y(x)), \quad y(x_0) = y_0 \qquad (y : \mathbb{R} \to \mathbb{R}^n,\ f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n)$$

numerically in some interval $x \in [x_0, x_{\text{end}}]$. For this purpose, use the embedded Runge-Kutta-Fehlberg scheme of order 2(3)

$$y_1 = y_0 + h \sum_{i=1}^{3} b_i k_i \quad \text{and} \quad \hat{y}_1 = y_0 + h \sum_{i=1}^{4} \hat{b}_i k_i$$

with increments

$$k_i = f\left( x_0 + c_i h, y_0 + h \sum_{j=1}^{i-1} a_{ij} k_j \right) \quad \text{for} \ \ i = 1, \ldots, 4.$$

The coefficients of this explicit method read:

| | | | | |
|---|---|---|---|---|
| $c_1$ | | | | |
| $c_2$ | $a_{21}$ | | | |
| $c_3$ | $a_{31}$ | $a_{32}$ | | |
| $c_4$ | $a_{41}$ | $a_{42}$ | $a_{43}$ | |
| | $b_1$ | $b_2$ | $b_3$ | |
| | $\hat{b}_1$ | $\hat{b}_2$ | $\hat{b}_3$ | $\hat{b}_4$ |

$\leftrightarrow$

| | | | | |
|---|---|---|---|---|
| 0 | | | | |
| 1/4 | 1/4 | | | |
| 27/40 | -189/800 | 729/800 | | |
| 1 | 214/891 | 1/33 | 650/891 | |
| | 214/891 | 1/33 | 650/891 | |
| | 533/2106 | 0 | 800/1053 | -1/78 |

Remark that it holds $c_4 = 1$ and $a_{4j} = b_j$ for $j = 1, 2, 3$. The approximation $y_1$ shall be the initial value of the next step if the current step is accepted. This means, one function evaluation of $f$ can be saved in case of an accepted step. Include this strategy in your implementation.

**Task 1:** Implement a MATLAB M-file with the following input and output arguments:

```
function [x,y,istat,idid] = rkf23(fname,xspan,y0,rtol,atol,h0,const)
% Runge-Kutta-Fehlberg 2(3)
%
% Input parameters:
%  fname        name of right-hand side f
%  xspan        vector [x0,xend]
%  y0           intial value (dimension n*1)
%  rtol         relative tolerance
%  atol         absolute tolerance
%  h0           initial step size
%  const = 0:   use step size control
%        = 1:   use constant step size h0
%
% Output parameters:
%  x            vector with points of independent variable
%  y            matrix with approximations at points x
  istat        statistics
               istat = [number of evaluations of right-hand side f,
               number of accepted steps, number of rejected steps ]

  idid = 1:   xend was reached
       = 0:   step size too small
```

The method should be implemented including step size control (which can be turned off with the flag `const = 1`). For that, use the estimated local error

$$e := y_1 - \hat{y}_1 = \mathcal{O}(h^3)$$

and the formula for step size prediction

$$h_{\text{opt}} = h_{\text{used}} \cdot \sqrt[3]{\frac{1}{\text{ERR}}}$$

with the error norm

$$\text{ERR} := \sqrt{\frac{1}{n}\sum_{j=1}^{n}\left(\frac{e_j}{\text{atol} + z_j \cdot \text{rtol}}\right)^2}, \quad \text{where } z_j := \max\{|y_{0,j}|,\, |y_{1,j}|\}\,.$$

Consequently, the step is accepted if $\text{ERR} \leq 1$ holds.

Include the safety factor $\rho = 0.9$ for scaling the resulting $h_{\text{opt}}$ and bounds $\theta = 5$, $\sigma = 0.2$ for increasing/decreasing the step size to avoid oscillating behavior (cf. lecture notes).

If the step size becomes smaller than $h_{\min} := 10^{-6}$, integration shall be terminated with the according information in `idid`.

**Task 2:** MATLAB provides already built-in one-step solvers with step size prediction.
Have a look in the MATLAB documentation to get familiar with the routines `ode23` and `ode45`.
Its options can be set via the command `odeset`.

**Remarks:**

- All built-in MATLAB routines are documented. You can have a look at the documentation via the commands `doc` or `help`, e.g. with `help ode23`.
- The dimension $n$ of the ODE system can be determined inside `rkf23` by the initial vector `y0` using the command `length` or `size`.
- Each right-hand side $f$ has to be implemented in an own M-file `fname.m` as a function of the form: `function f = fname(x,y)`.
- The evaluation of the varying function `fname` inside `rkf23` can be done using the command `feval`, for example via: `f = feval(fname,x,y)`.
- If the right-hand side $f$ is given in the file `fname.m`, then the integrator is called by: `[...]=rkf23(@fname,...)` or `[...]=rkf23('fname',...)`.
- The size of the output parameters `x` and `y` is not known a priori. The vector and the matrix can be enlarged in each step. However, this extension may take a long time for larger sizes. At the beginning, fix `x` and `y` using the command `zeros` to a maximum number of steps, say 10000. Cut off the unused part at the end of the integration.

For further information on the lab exercises, see:
`http://www-num.math.uni-wuppertal.de/en/amna/teaching/lectures/`
`lab-exercises-for-numerical-analysis-and-simulation-i-odes.html`

**Test examples:**

**a)** *Scalar ODE*

The initial value problem

$$y'(x) = \frac{y}{1+x^2}, \qquad y(x_0) = y_0$$

exhibits the exact solution

$$y(x) = y_0 \cdot \mathrm{e}^{\arctan(x)-\arctan(x_0)}.$$

Solve the problem in the interval $x \in [-10, 20]$ with initial condition $y(-10) = 1$. Apply a step size control with the initial step size $h_0 = 1$. Choose the fixed absolute tolerance $atol = 10^{-6}$, whereas the relative tolerances $rtol = 10^{-l}$ for $l = 4, 6$ should be applied. Have a look at the statistics: how many function evaluations are needed?
Try also the methods `ode23` and `ode45` as well as the method with constant step sizes in the case $h = 0.1$. Compare the seven numerical approximations to the exact solution.

**b)** *Van-der-Pol Oscillator*

The second order ODE

$$y'' = \mu(1-y^2)y' - y$$

describes the Van-der-Pol Oscillator. Apply the equivalent system of first order. Consider the initial values $y(0) = 2,\ y'(0) = 0$.

Let $atol = rtol = 10^{-2}$, $h_0 = 10^{-2}$ and $x_{\mathrm{end}} = 30$ be fixed. Solve the IVP for the different parameters $\mu = 1, 10, 100$ with `rkf23` and discuss the development of the solution and the step size control. Try also the methods `ode23` and `ode45`.

**c)** *Three-body problem*

We consider a three-body problem with the earth, the moon and a satellite in a two-dimensional space $(\xi_1, \xi_2)$. The unknowns are the position and velocity of the satellite: $y_1 := \xi_1, y_2 := \xi_2, y_3 := \xi_1', y_4 := \xi_2'$. The system of ODEs reads

$$\begin{aligned}
y_1'(x) &= y_3 \\
y_2'(x) &= y_4 \\
y_3'(x) &= y_1 + 2y_4 - (1-\mu)\tfrac{y_1+\mu}{r_1^3} - \mu\tfrac{y_1-1+\mu}{r_2^3} \\
y_4'(x) &= y_2 - 2y_3 - (1-\mu)\tfrac{y_2}{r_1^3} - \mu\tfrac{y_2}{r_2^3}
\end{aligned}$$

with the parameter $\mu = \frac{1}{82.45}$ and

$$r_1 := \sqrt{(y_1+\mu)^2 + {y_2}^2}, \qquad r_2 := \sqrt{(y_1-1+\mu)^2 + {y_2}^2}.$$

Solve the system with the initial values

$$y(0) = (1.2, 0, 0, -1.049358)^\top$$

in the time interval $x \in [0, T]$ with $T := 6.1921693$. Thereby, select the initial step size $h_0 = T/1500$. Apply constant step sizes as well as step size control with the tolerances $rtol = atol = 10^{-6}$. Plot the resulting satellite's orbits in $(\xi_1, \xi_2)$ phase diagrammes. Compare the results of the two simulations.