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Bergische Universität Wuppertal
Fachbereich C – Mathematik und Naturwissenschaften
Angewandte Mathematik / Numerische Analysis

Numerical Analysis and Simulation I: Ordinary Differential Equations (ODEs)

Sheet 12 - Boundary value problems

Return of the Exercise Sheet: Tuesday, January 27th before the lecture
Examination: Tuesday, February 10, 10:00-12:00 in Hörsaal 4 (F.10.01)

Exercise 26: *Single Shooting Method*
Consider the boundary value problem (BVP)

$$u'' + u = 0, \qquad 0 < x < 1, \qquad u(0) = 0, \quad u(1) = 1. \tag{BVP}$$

a) Check that $u(x) = \sin(x)/\sin(1)$ is an exact solution of the problem (BVP).

b) Rewrite the boundary value problem (BVP) as a system of first order equations.

c) Formulate the single shooting method for (BVP).

d) Formulate the Newton method to solve the nonlinear system appearing in b).
State explicitly the Jacobian used in the Newton method.

e) Considering the exact solution $u(x)$, compute the exact value of the shooting parameter $\tilde{s}$!

Exercise 27: *Initial and boundary value problem*
For the linear two-point boundary value problem

$$y''(x) + y(x) = 0, \qquad (y : \mathbb{R} \to \mathbb{R})$$

we define the following different boundary conditions

(i) $y(0) = y(\pi/2) = 1$,

(ii) $y(0) = y(\pi) = 1$,

(iii) $y(0) = y(2\pi) = 1$.

a) Solve the differential equation with the initial values $y(0) = a,\ y'(0) = b$.

b) Discuss the solution for the different boundary value problems.

c) Compare the existence and uniqueness for initial and boundary value problems.

Homework 23: *Single Shooting Method* (10 Points)
We consider the implicitly given second order differential equation

$$xy''(x) - 2xy'(x) + 2y = x^3 \sin(x), \quad a := \frac{\pi}{2} \le x \le 2\pi =: b$$

together with the boundary conditions

$$y(a) = \alpha, \quad y(b) = \beta,$$

where

$$\alpha = \frac{1}{20}\pi^2 - \pi, \quad \beta = \frac{4}{5}\pi^2 - 2\pi.$$

a) Transform the given differential equation into an explicit first order system of ODEs.

b) In order to use the single shooting method, determine the algebraic equation $g : \mathbb{R} \to \mathbb{R}$ for the unknown initial value $\gamma = y'(a)$.

c) Use the bisection method to find the root of g. Start with the interval where the boundaries are located at $\gamma_L = \frac{1}{5}\pi - 5$ and $\gamma_R = \frac{1}{5}\pi + 3$, where it holds $g(\gamma_L) < 0$ and $g(\gamma_R) > 0$.

Hint: It holds
$$\frac{-4}{\pi^2}\begin{pmatrix} 1 & -\pi/2 \\ -\pi & \pi^2/4 \end{pmatrix}\begin{pmatrix} \frac{1}{20}\pi^2 - \frac{\pi}{2} \\ \frac{1}{5}\pi - k + 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{5} + \frac{4-2k}{\pi} \\ k - 3 \end{pmatrix}.$$

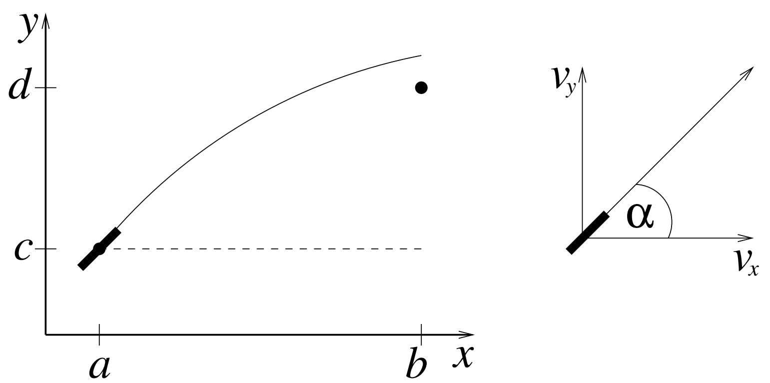
d) State the solution of the boundary value problem.

Remark: In order to avoid the application of numerical methods for solving the differential equation in each step, we have chosen an ODE where the analytical solution is known in this exercise. You may use the following formula to evaluate the solution of the ODE in the final point $x = b$:

$$y(x) = C_1x^2 + C_2x - x\sin(x), \quad \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \frac{-4}{\pi^2}\begin{pmatrix} 1 & -\pi/2 \\ -\pi & \pi^2/4 \end{pmatrix}\begin{pmatrix} \alpha + \pi/2 \\ \gamma + 1 \end{pmatrix},$$

with $\alpha = y(a)$ and $\gamma = y'(a)$.

Homework 24: *Boundary value problems for shooting cannon* (10 Points)
In this exercise, the formulation of boundary value problems is discussed using an example from mechanics. Numerical methods are not considered here.



A cannon is located at $x = a, y = c$. We want to hit the position $x = b, y = d$, where $a < b$ and $c < d$. The dynamics of the fired ball is described by the system of ODEs

$$x'(t) = v_x(t), \qquad y'(t) = v_y(t), \qquad v_x'(t) = 0, \qquad v_y'(t) = -g$$

with the gravitation constant $g > 0$. We fire the cannon at time $t = 0$. The ball leaves the cannon with velocity $v_0 = \sqrt{v_x(0)^2 + v_y(0)^2}$. Let α be the angle between the cannon and the horizontal axis.
Attention: The independent variable is t. The space coordinates x, y represent dependent variables. Boundary value problems are considered in a time interval $t \in [0, T]$.

a) Assume that the final time $T > 0$ is predetermined, when the ball reaches the desired position. Formulate a two point boundary value problem for the system of ODEs. Determine the initial values $x(0), y(0), v_x(0), v_y(0)$ in dependence on T.

b) Let the initial speed v_0 of the ball be a given constant. Thus we can choose the angle α only. Determine the initial values $x(0), y(0), v_x(0), v_y(0)$ and a corresponding final time $T > 0$ such that the boundary value problem is satisfied. Arrange a formula for the required angle α.