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Fachbereich C – Mathematik und Naturwissenschaften
Angewandte Mathematik / Numerische Analysis

Numerical Analysis and Simulation I: Ordinary Differential Equations (ODEs)

Sheet 10 - Geometric Integrators and Stochastic Differential Equations

Return of the Exercise Sheet: Tuesday, January 13th before the lecture

Exercise 21: *Toda-Flow with Matrix-Differential Equation*
We consider the solution of the matrix differential equation

$$Q' = Q \cdot B, \qquad \text{with } Q(0) = I,$$

where $B \in \mathbb{R}^{m \times m}$ is a skew-symmetric matrix, i.e. $B^T = -B$ (and I is the identity map of $\mathbb{R}^m$). Furthermore, let $\{Q_n\}_{n=0}^{\infty}$ be the numerical approximation produced by an Runge-Kutta scheme.
Now, prove the following: If the RK-coefficients fulfill

$$m_{ij} = b_i a_{ij} + b_j a_{ji} - b_i b_j = 0 \qquad for\, 1 \leq i, j \leq s,$$

then Q_n is orthogonal (i.e. $Q_n \cdot Q_n^T = I = Q_n^T \cdot Q_n$) for all $n \geq 0$.

Hint:

- use the internal stages $U_i = Q_n + h \sum_{j=1}^{s} a_{ij} K_j$ with $K_i = U_i \cdot B$ to set up the Runge-Kutta scheme
- prove $Q_n Q_n^\top = I$ by induction over $l \geq 0$:
  - start with the base case $l = 0$ inductive step
  - inductive step $n \mapsto n + 1$:
    use the Runge-Kutta approximation for Q_{n+1},
    replace Q_n in a second step and combine the coefficients
- deduce from this result $Q_n^\top Q_n = I$

Exercise 22: *Stochastic Differential Equation*
Let X_t be an Itô-process, satisfying

$$dX_t = \mu X_t \, dt + \sigma X_t \, dW_t.$$

We define a new stochastic process by $Y_t := X_t^\alpha$, $\alpha \in \mathbb{R}$.
Determine the stochastic differential equation (SDE) for Y_t.

Homework 19: *Determinant as First Integral* (10 Points)
Consider the quasi-linear problem

$$Y' = A(Y) \cdot Y \;, \qquad Y(0) = Y_0$$

with Y and $A(Y)$ being matrices of size $n \times n$. If the trace of $A(Y)$ is equal to zero, it can be shown that the determinant is an invariant of the matrix differential equation $Y' = A(Y) \cdot Y$.

a) Let $g(Y) := \det Y$ be the determinant of Y. Show that the Abel-Liouville-Jacobi-Ostrosgraskii identity

$$g'(Y) \cdot A(Y) \cdot Y = \operatorname{trace} A(Y) \cdot \det Y$$

is fulfilled if $\operatorname{trace} A(Y) = 0$ for all Y.

Hint: Start with the development of a Taylor series of Y around $x_0 = 0$ and then use the Laplace expansion theorem.

b) Prove that the determinant is a first integral of the matrix differential equation $Y' = A(Y)Y$ if $\operatorname{trace} A(Y) = 0$ for all Y.

Homework 20: *Wiener-Process* (10 Points)
Let W_t be a Wiener-process. Show that it holds

$$E\big(W_t^4\big) = 3t^2.$$

Hint: Use the lemma of Itô on $Z_t = W_t^4$. Moreover, it holds $\int_0^t W_\tau^3 dW_\tau = 0$.