Prof. Dr. Michael Günther
Zuzana Bučková, M.Sc.
Dmitry Shcherbakov, M.Sc.
Dipl. Math. Michèle Wandelt, M.Sc.
Winter Term 2014/15

Bergische Universität Wuppertal
Fachbereich C – Mathematik und Naturwissenschaften
Angewandte Mathematik / Numerische Analysis

Numerical Analysis and Simulation I: Ordinary Differential Equations (ODEs)

Sheet 7 - Differential Algebraic Equations

Return of the Exercise Sheet: Tuesday, December 9th before the lecture

Exercise 14: *DAE-index*
Let $f, g : \mathbb{R}^2 \to \mathbb{R}$ be smooth functions. Consider the semi-explicit DAE

$$\begin{aligned} u'(x) &= f(u(x), v(x)) + \lambda(x)u(x), \\ v'(x) &= g(u(x), v(x)) + \lambda(x)v(x), \\ 1 &= u^2(x) + v^2(x), \end{aligned}$$

with solutions $y(x) = \big(u(x), v(x), \lambda(x)\big)^\top \in \mathbb{R}^3$.

a) Show that the differential index of this DAE is 2.

b) How many free initial conditions are there? Which conditions have to be fulfilled for an initial value such that an IVP has a solution?

c) Now let $f(u, v) \equiv 0$, $g(u, v) \equiv -5$ and $(u_0, v_0, \lambda_0)^\top = (1, 0, 0)^\top$. Formulate the system that has to be solved for the first step of the implicit Euler scheme for this special case.

d) Consider the conditions $f(u, v) \equiv 0$, $g(u, v) \equiv -5$ mentioned in part c) again.

- Is there an exact solution with the initial value $(u(0), v(0), \lambda(0))^\top = (1, 0, 0)^\top$? Why?
- Is there also an exact solution with the initial value $(\tilde{u}(0), \tilde{v}(0), \tilde{\lambda}(0))^\top = (0, 1, 0)^\top$? Why?

Exercise 15: *Index in Example by Campbell/Gear*
We investigate the example of Campbell and Gear (1995):

$$y_3 N\dot{y} + y = 0, \qquad y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, \qquad N = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Due to $\det(N) = 0$, a quasilinear system of DAEs arises. Determine the (unique) solution.
Show that the differential index of the system is $k = 1$, whereas the perturbation index of the system results to $k = 3$.
Hint: To determine the perturbation index, investigate the differences $\hat{y}_i - y_i$ for $i = 1, 2, 3$ and observe which derivatives of the perturbation appear. This already indicates the perturbation index. The determination of an upper bound for $\|\hat{y} - y\|$ is not necessary.
Generalise to a system with m equations, where the differential index remains $k = 1$ and the perturbation index becomes $k = m$.

Exercise 16: *Solutions of Linear DAEs. (just to get familiar with the topic)*
Determine the solution of the following linear systems of DAEs (with non-constant matrices). Investigate which initial values $y(x_0) = y_0$ are feasible.
Hint: Observe the two equations of each system. Try to eliminate one unknown (y_1 or y_2) to obtain a scalar ODE, which is solved analytically.

a) $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \mathrm{e}^x + \cos(x) \\ \mathrm{e}^x \end{pmatrix}$

b) $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ -x & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}$

Homework 13: *DAE-index* (10 Points)
Let C, R, a, ω be positive real parameters. A simple circuit is modelled by the linear DAE

$$\begin{aligned} -C(v-u)' + i &= 0 \\ C(v-u)' - \tfrac{1}{R}v &= 0 \\ u &= a\sin(\omega x) \end{aligned} \qquad \text{with solutions } y(x) = \begin{pmatrix} u(x) \\ v(x) \\ i(x) \end{pmatrix} \in \mathbb{R}^3, x \in [0, b].$$

a) Write the system in the form $My' = f(x, y(x))$, i.e. give M and f. What is the rank of M?

b) How many free initial conditions are there? Which conditions have to be fulfilled for an initial value $y_0 = \begin{pmatrix} u_0 \\ v_0 \\ i_0 \end{pmatrix}$ such that an IVP has a solution?

c) Show that the differential index of this DAE is 1.

d) Let the initial value $y_0 = \begin{pmatrix} u_0 \\ v_0 \\ i_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ be given.
Formulate the system that has to be solved for the first step $y_1 \approx y(h)$ of the implicit Euler scheme for a given stepsize h.

e) Explain shortly why explicit schemes do not work here.

Homework 14: *Differential Index* (10 Points)
We consider the differential index for DAEs of different shapes. Prove the following:

a) The system

$$\begin{aligned} y' &= f(y, z)\ , \\ 0 &= g(y, z) \end{aligned}$$

has differential index $k = 1$ if g_z is regular.

b) The system

$$\begin{aligned} y' &= f(y, z)\ , \\ 0 &= g(y) \end{aligned}$$

has differential index $k = 2$ if f_z and g_y are regular.