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Fachbereich C – Mathematik und Naturwissenschaften
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Numerical Analysis and Simulation I: Ordinary Differential Equations (ODEs)

Sheet 6 - Stiff Systems and Stability

Return of the Exercise Sheet: Tuesday, December 2nd before the lecture

Exercise 12: *Stability functions for Runge-Kutta schemes*
An implicit Runge-Kutta scheme (IRK) is given in the form

$$\begin{aligned} y_1 &= y_0 + h\sum_{i=1}^{s} b_i k_i \\ k_i &= \lambda u_i \\ u_i &= y_0 + h\sum_{j=1}^{s} a_{ij} k_j \ . \end{aligned}$$

We investigate the stability function $R : \mathbb{C} \to \mathbb{C}$ for Runge-Kutta schemes considering Dahlquist's test equation

$$y' = \lambda y, \quad y(0) = y_0, \quad \lambda \in \mathbb{C} \quad \text{with } \operatorname{Re}(\lambda) \le 0$$

for $y : \mathbb{R} \to \mathbb{C}$.

Let $A = (a_{ij}) \in \mathbb{R}^{s\times s}$ and $b = (b_i) \in \mathbb{R}^s$ contain the coefficients of the scheme. We define $z := h\lambda$.

a) Show that the stability function can be written as

$$R(z) = \frac{\det(I - zA + z\mathbb{1}b^\top)}{\det(I - zA)}$$

such that $y_1 = R(z)\, y_0$ holds. (The matrix $I - zA$ is non-singular for almost all $z \in \mathbb{C}$.)

Hint: Use a vector notation with

$$K := (k_1, \dots, k_s)^\top, \qquad U := (u_1, \dots, u_s)^\top, \qquad \mathbb{1} := (1, \dots, 1)^\top$$

and apply Cramer's rule for linear systems.

b) Show that

$$R(z) = 1 + zb^\top (I - zA)^{-1}\mathbb{1} \ .$$

Hint: You do not need to apply the result of part (a) here.

c) For an explicit Runge-Kutta method, the matrix A is nilpotent. In this case, prove the relation

$$R(z) = 1 + zb^\top \left(\sum_{j=0}^{s-1} (zA)^j \mathbb{1} \right) \ .$$

Consequently, the stability function $R(z)$ is a polynomial in z of degree $\le s$!
Is it possible that the method is A-stable?

Hint: Examine $\lim R(z)$ for $\operatorname{Re}(z) \to -\infty$.

Exercise 13: *Stability Functions for One Step Methods*
In this exercise, we consider four numerical one step methods

a) explit Euler scheme, $y_1 = y_0 + hf(x_0, y_0)$

b) implicit Euler scheme, $y_1 = y_0 + hf(x_1, y_1)$

c) trapezoidal rule, $y_1 = y_0 + h\left(\frac{1}{2}f(x_0, y_0) + \frac{1}{2}f(x_1, y_1)\right)$

d) implicit mid-point rule, $y_1 = y_0 + hk_1,\ k_1 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2}k_1\right)$.

Apply Dahlquist's test equation to each of these methods and thus compute the stability function $R(z)$. Also sketch the domain of stability. For each method, make a statement regarding A- and L-stability, respectively.

Homework 11: *Diagonal Implicit Runge-Kutta methods* (10 Points)
Let an initial value problem $y' = f(x, y),\ y(x_0) = y_0\ (y : \mathbb{R} \to \mathbb{R}^n)$ be given. Consider the diagonal implicit Runge-Kutta (DIRK) methods specified by the Butcher tableaus:

$$\text{(i)}\quad \begin{array}{c|cc} \frac{1}{4} & \frac{1}{4} & 0 \\ \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array} \qquad\qquad \text{(ii)}\quad \begin{array}{c|cc} 0 & 0 & 0 \\ \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ \hline & \frac{1}{4} & \frac{3}{4} \end{array}$$

The scheme (i) is a singly diagonal implicit Runge-Kutta (SDIRK) method of order 3 and the scheme (ii) is the Hammer & Hollingsworth method of order 3.

a) Determine the stability function $R : \mathbb{C} \to \mathbb{C}$ for both methods considering Dahlquist's test equation. Which part of the real axis belongs to the stability domain? Are these schemes A-stable or L-stable? Why?

Hint: $R(z) = \det(I - zA + z\mathbb{1}b^\top)/\det(I - zA)$

b) How many (sub-)systems of dimension n have to be solved in a (simplified) Newton iteration for (i) and (ii), respectively?

Homework 12: *Dissipative Systems* (10 Points)
Prove the following:

a) The test equation

$$y' = \lambda y$$

is contractive (or dissipative) for $\Re(\lambda) \leq 0$, where $\lambda, y \in \mathbb{C}$.

Hint: Transform $y \in \mathbb{C} \to \mathbb{R}^2$.

b) A B-stable one-step method is also A-stable.