Prof. Dr. Michael Günther
Zuzana Bučková, M.Sc.
Dmitry Shcherbakov, M.Sc.
Dipl. Math. Michèle Wandelt, M.Sc.
Winter Term 2014/15

Bergische Universität Wuppertal
Fachbereich C – Mathematik und Naturwissenschaften
Angewandte Mathematik / Numerische Analysis

Numerical Analysis and Simulation I: Ordinary Differential Equations (ODEs)

Sheet 3 - Consistency and Runge-Kutta Methods

Return of the Exercise Sheet: Tuesday, November 11th before the lecture

Exercise 5: *Construction of Runge-Kutta methods*
For an explicit Runge-Kutta method with two stages

$$y(x+h) \doteq y_0 + h\sum_{i=1}^{s} b_i k_i$$

$$k_i = f(x_0 + c_i h, y_0 + h\sum_{j=1}^{s} a_{ij}k_j), \quad i = 1, \ldots, s$$

(which not necessary assumes the nodes relation $c_i = \sum_j a_{ij}$, for all i), we have the following coefficients: the nodes c_1, c_2, the weights b_1, b_2 and the sole non-vanishing entry a_{21} of matrix A.

a) State the corresponding Butcher tableau.

b) Which equations have to be posed on the five parameters c_1, c_2, b_1, b_2 and a_{21}, such that the resulting (explicit RK-) method is of order two?

c) Construct methods of order two, for which it holds

(i) $c_1 = 0,\ b_1 = 0\,;$

(ii) $c_1 = 0,\ b_1 = 1/2\,;$

(iii) $c_1 = 1,\ b_1 = 1/2\,.$

Give the resulting increment function $\Phi(x, y, h; f)$ for an initial value problem $y' = f(x, y)$, with $y(0) = y_0$.

Exercise 6: *Consistency of the trapezoidal rule*
We consider the initial value problem of an autonomous ODE

$$y'(x) = f(y), \quad y(x_0) = y_0 \qquad (y : \mathbb{R} \to \mathbb{R})$$

with $f \in C^3$ and assume that the solution satisfies $y \in C^4$. The trapezoidal rule implies the numerical technique

$$y_1 = y_0 + \tfrac{h}{2}\left[f(y_0) + f(y_1)\right].$$

The defect $\delta(h)$ of this scheme is defined by using the exact solution

$$\delta(h) := y(x_0 + h) - y(x_0) - \tfrac{h}{2}\left[f(y(x_0)) + f(y(x_0 + h))\right].$$

a) Proof that $\delta(h) = \mathcal{O}(h^3)$ holds and calculate the dominating term explicitly.

b) Let f fulfill the global Lipschitz condition $|f(v) - f(w)| \leq L|v - w|$ for all v, w.
Use the defect to obtain an estimate for the local discretisation error, which shows that the method is consistent of order two.

Homework 5: *Order barriers for explicit Runge-Kutta schemes* (10 Points)
Consider an explicit Runge-Kutta scheme with s stages described by the Butcher tableau

$$\begin{array}{c|cccc} c_1 & 0 & \dots & \dots & 0 \\ c_2 & a_{21} & 0 & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ c_s & a_{s1} & \dots & a_{s,s-1} & 0 \\ \hline & b_1 & b_2 & \dots & b_s \end{array} \qquad \text{with} \qquad c_i = \sum_{j=1}^{i-1} a_{ij} \quad \text{for } i = 1, \dots, s.$$

Such a scheme can not exhibit the order $p = s + 1$.

a) Prove the statement for the cases $s = 1, 2, 3$.

b) Let the coefficients a_{ij} be collected in the matrix $A \in \mathbb{R}^{s\times s}$ and the coefficients c_i, b_i in the column vectors $c,\ b \in \mathbb{R}^s$. For arbitrary $p \geq 2$, a specific order condition reads

$$b^\top A^{p-2} c = \frac{1}{p!}.$$

Find a proof for arbitrary s based on the structure of A and c.

Homework 6: *Consistency of trapezoidal rule (part II)* (10 Points)
We consider the initial value problem of an autonomous ODE

$$y'(x) = f(y)\,, \qquad y(x_0) = y_0 \qquad (y : \mathbb{R} \to \mathbb{R})\,,$$

with $f \in C^3,\ y \in C^4$. The trapezoidal rule implies the numerical technique

$$y_1 = y_0 + \tfrac{h}{2}\left[f(y_0) + f(y_1)\right].$$

We assume that $y_1 = y_0 + \mathcal{O}(h)$ and $f(y_1) = f(y_0) + \mathcal{O}(h)$ holds (satisfied, for example, if f is bounded and fulfills the Lipschitz condition).
Prove that the numerical method is consistent of order 2 by calculating the dominating term of the local discretization error

$$\tau(h) := \frac{y(x_0 + h) - y_1}{h}$$

explicitly.

Hint: Insert the complete formula for y_1 in $f(y_1)$. Perform a Taylor expansion of $f(y_1)$. Insert the complete Taylor expansion of $f(y_1)$ in one term of its own right-hand side.